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VIBRATION AND STABILITY OF THICK SIMPLY SUPPORTED SHALLOW SHELLS SUBJECTED TO IN-PLANE STRESSES

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The effects of higher order deformations on natural frequencies and buckling stresses of a thick shallow shell with reactangular planform subjected to uniaxial and biaxial in-plane stresses are studied. Based on the power series expansion of displacement components, a set of fundamental dynamic equations of a two-dimensional higher order shallow shell theory is derived through Hamilton's principle. Several sets of truncated approximate theories which can take into account the complete effects of higher order deformations such as shear deformations with thickness changes and rotatory inertia are applied to solve the vibration and stability problems of a thick shallow shell. Three types of simply supported shallow shells with positive, zero and negative Gaussian curvatures are considered. In order to assure the accuracy of the present theory, convergence properties of the lowest two natural frequencies for the first vibration mode $r = s = 1$ are examined in detail. The present results are also compared with those of existing theories. In the case of a simply supported shallow shell, buckling stresses can be calculated from the natural frequencies without in-plane stresses.

1. INTRODUCTION

A number of significant contributions have been made for vibration and stability problems of shells, but most of them deal with closed shells having various shapes and open shells have received little attention to date. In a review article on recent research advances in vibration of thin and thick shadow shells, attention has been given to the wide coverage of the classical, first order and higher order shallow shell theories by Liew *et al.* [1]. Reference has also been made to three-dimensional elasticity analyses of moderately thick shells. The practical importance of vibration and stability analyses of singly curved and/or doubly curved shallow shells with rectangular planform has been increased in structural, aerospace and mechanical engineering applications. For thin shells, almost all the analyses of shells were based on the classical theory, such as the Kirchhoff–Love theory which ignored the transverse shear deformations and thickness changes. The free flexural vibration of singly curved thin shallow shells of rectangular planform has been reported by Lim

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and Liew [2]. Based on the thin shallow shell theory, the *pb*-2 Ritz energy based approach was employed by introducing a product of two-dimensional orthogonal polynomials and a basic function for various boundary conditions into three displacements. Recently, Liew and Lim [3] investigated the free flexural vibration of thin doubly curved shells of rectangular planform by the same method. Shells of positive, zero and negative Gaussian curvatures with six different combinations of boundary conditions were analyzed. Although the vibration problems of thin elastic shallow shells have been investigated extensively, much attention is now being paid to thick shallow shells. The classical thin shallow shell theory has been known to be inadequate for the vibration and buckling problems of higher modes and also inadequate for thick shells when the effects of shell thickness on frequencies and buckling stresses cannot be neglected. For thick concrete shallow shells in architectural roof structures prestressing stress is often introduced in the in-plane directions of the shell to avoid tensile stresses. The distribution of the prestressing stress in the shell section is controlled through the prestressing strands. In order to predict accurately the dynamic characteristics of thick shells subjected to such prestressing stresses, more refined theories which are to incorporate the effects of higher order deformations such as shear deformations with thickness changes are needed. The effects of shear deformations and thickness changes have been shown to influence the natural frequencies and buckling stresses of deep beams [4] and thick plates [5] significantly. The same feature can be said about thick shallow shells to which the classical shell theory is no longer applicable. In order to incorporate the effects of shear deformations and thickness changes in the plate and shell problems, two methods of analysis have been used, i.e., one is based on the three-dimensional elasticity theory and the other, approximate two-dimensional *shell* theory. The three-dimensional Ritz approach has been developed on the basis of a three-dimensional elasticity theory for the analysis of vibration of thick plates [6]. For the free vibration problem of a thick cylindrical shell or panel, a three-dimensional solution method has been presented by Soldatos and Hadjigeorgiou [7]. The governing equations of three-dimensional linear elasticity were solved by using an iterative mathematical approach to obtain the natural frequencies of a simply supported cylindrical shell.

By expanding the shell displacement components in power series of the thickness co-ordinate, there exist approximate two-dimensional shell theories. Upon using certain truncations of the power series, approximate shell theories which can take into account the first order effects of transverse shear deformations have been applied to cylindrical shells [8, 9]. Based on the first order and higher order shallow shell theories, Ritz vibration analyses of moderately thick shallow shells have been presented by Liew and Lim [10–13]. Since the normal displacement was assumed to be constant through the thickness of a shell, the thickness change of the shell is not allowed and the normal strain in the thickness direction vanishes. For moderately thick and thick shallow shells, the effects of thickness changes as well as shear deformations should be taken into account. However, two-dimensional higher order theories of shallow shells which take into account the complete effects of shear deformations with thickness changes and rotatory inertia have not been investigated.

This paper presents a two-dimensional higher order theory of thick shallow shells which can take into account the complete effects of both shear deformations with thickness changes and rotatory inertia. Several sets of the governing equations of truncated approximate theories are applied to the analysis of vibration and stability problems of a shallow shell subjected to in-plane stresses. On the basis of the power series expansions of displacement components, a fundamental set of dynamic equations of a two-dimensional higher order theory for the vibration problem of thick shallow shells is derived through Hamilton's principle. The equations of motion of a shell subjected to in-plane stresses are expressed in terms of the displacement components. Following the Navier solution procedure, the displacement components are expanded into Fourier series that satisfy the simply supported boundary conditions. Three types of simply supported shallow shells with positive, zero and negative Gaussian curvatures are considered. Natural frequencies of a thick shallow shell subjected to in-plane stresses are obtained by solving the eigenvalue problem numerically. When the natural frequency vanishes under the in-plane stresses, elastic buckling occurs and the critical buckling stress relates with the natural frequency of the shell without in-plane stresses. The critical buckling stress of simply supported shallow shells under uniaxial and biaxial in-plane stresses can be predicted from the natural frequency of the shell without in-plane stresses. The convergence properties of the present numerical solutions are shown to be accurate for the natural frequencies and buckling stresses with respect to the order of approximate theories. A comparison of the obtained natural frequencies is also made with those of existing theories such as the classical shell theory and the first order shear deformation shell theory. The present results obtained by various sets of approximate theories are considered to be accurate enough for thick shallow shells. It is noticed that the two-dimensional higher order shallow shell theory in the present paper is useful for vibration and stability problems of extremely thick shallow shells.

2. FUNDAMENTAL EQUATIONS OF KINEMATICS OF SHALLOW SHELLS

Introducing a curvilinear co-ordinate system x^α $(\alpha = 1, 2)$, x^3 on the middle surface of a shell of uniform thickness h, the dynamic displacement components in a shell are expressed as

$$v_\alpha \equiv v_\alpha(x^\alpha, x^3; t), \qquad v_3 \equiv v_3(x^\alpha, x^3; t), \tag{1}$$

where t denotes time. The displacement components may be expanded into a power series of the thickness co-ordinate x^3 as follows:

$$v_\alpha = \sum_{n=0}^{\infty} \overset{(n)}{v}_\alpha (x^3)^n, \qquad v_3 = \sum_{n=0}^{\infty} \overset{(n)}{v}_3 (x^3)^n, \tag{2}$$

where $n = 0, 1, 2, \ldots, \infty$.

Based on this expression of the displacement components, a set of linear fundamental equations of a two-dimensional higher order theory for a thick shallow shell can be summarized in the following.

2.1. STRAIN–DISPLACEMENT RELATIONS

Strain components may be expanded as follows:

$$\gamma_{\alpha\beta} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha\beta}(x^3)^n, \qquad \gamma_{\alpha 3} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha 3}(x^3)^n, \qquad \gamma_{33} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{33}(x^3)^n. \tag{3}$$

The expanded linear strain–displacement relations of a shallow shell within the assumption of $h/R \ll 1$ (where R is the least principal radius of curvature) can be written as [14]

$$\overset{(n)}{\gamma}_{\alpha\beta} = \tfrac{1}{2}(\overset{(n)}{v}_{\alpha,\beta} + \overset{(n)}{v}_{\beta,\alpha} - 2b_{\alpha\beta}\overset{(n)}{v}_3), \tag{4}$$

$$\overset{(n)}{\gamma}_{\alpha 3} = \tfrac{1}{2}\{(n+1)\overset{(n+1)}{v_\alpha} + \overset{(n)}{v}_{3,\alpha}\}, \qquad \overset{(n)}{\gamma}_{33} = (n+1)\overset{(n+1)}{v_3}, \tag{5}$$

where $b_{\alpha\beta}$ denotes the covariant component of curvature tensor of the shell middle surface. Greek lower case subscripts are assumed to range over the integers 1, 2. With the use of shallowness assumption of $(L/R)^2 \ll 1$ (where L denotes the wavelength of the deformation pattern), a comma indicates partial differentiation with respect to the curvilinear co-ordinate subscripts that follow.

2.2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Introducing stress components $s^{\alpha\beta}$, $s^{\alpha 3}$ and s^{33}, Hamilton's principle is applied to derive the equations of dynamic equilibrium and natural boundary conditions of a shell. In order to treat vibration and stability problems of a shell subjected to in-plane stresses $s_0^{\alpha\beta}$ which distribute uniformly in x^1 and/or x^2 directions and arbitrarily in the thickness direction, additional works due to these stresses which are assumed to remain unchanged during vibration and/or buckling are taken into consideration. Both the upper and lower surfaces of a shell are assumed to be traction free. The principle for the present problems may be expressed for an arbitrary time interval t_1 to t_2 as follows:

$$\int_{t_2}^{t_1}\int_V [s^{\alpha\beta}\delta\gamma_{\alpha\beta} + 2s^{\alpha 3}\delta\gamma_{\alpha 3} + s^{33}\delta\gamma_{33} - \rho(\dot{v}^\alpha\delta\dot{v}_\alpha + \dot{v}^3\delta\dot{v}_3)$$
$$+ s_0^{\alpha\beta}(v^\lambda_{,\alpha}\delta v_{\lambda,\beta} + v^3_{,\alpha}\delta v_{3,\beta})]\,\mathrm{d}V\,\mathrm{d}t = 0, \tag{6}$$

where the over dot indicates partial differentiation with respect to time and ρ denotes the mass density, and $\mathrm{d}V$, the volume element. The in-plane stress $s_0^{\alpha\beta}$ is assumed to be expanded as the following power series:

$$s_0^{\alpha\beta} = \sum_{\ell=0}^{\infty} \overset{(\ell)}{s}{}_0^{\alpha\beta}(x^3)^\ell, \tag{7}$$

where $\ell = 0, 1, 2, \ldots, \infty$.

By performing the variation as indicated in equation (6), the equations of motion are obtained as follows:

$$\delta \overset{(n)}{v}_{\beta}\!: \overset{(n)}{N}{}^{\alpha\beta}_{,\alpha} - n \overset{(n-1)}{Q}{}^{\beta} + \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \overset{(\ell)}{s}{}^{\alpha\lambda}_{0} \overset{(m)}{v}{}^{\beta}{}_{,\alpha\lambda} f(n+m+\ell+1) = \rho \sum_{m=0}^{\infty} f(n+m+1) \overset{(m)}{\ddot{v}}{}^{\beta}, \tag{8}$$

$$\begin{aligned}\delta \overset{(n)}{v}_{3}\!: b_{\alpha\beta} \overset{(n)}{N}{}^{\alpha\beta} - n \overset{(n)}{Q}{}^{\alpha}{}_{,\alpha} - n \overset{(n-1)}{T} + \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \overset{(\ell)}{s}{}^{\alpha\beta}_{0} \overset{(m)}{v}{}^{3}{}_{,\alpha\beta} f(n+m+\ell+1) \\ = \rho \sum_{m=0}^{\infty} f(n+m+1) \overset{(m)}{\ddot{v}}{}^{3},\end{aligned} \tag{9}$$

where $n, m = 0, 1, 2, \ldots, \infty$, and the following equation is defined as

$$f(k) \equiv \int_{-h/2}^{+h/2} (x^3)^{k-1} \, \mathrm{d}x^3 = \frac{1}{k}\left(\frac{h}{2}\right)^k [1 - (-1)^k] = \begin{cases} 0, & k \text{ is even}, \\ \dfrac{2}{k}\left(\dfrac{h}{2}\right)^k, & k \text{ is odd}, \end{cases} \tag{10}$$

where k is an integer.

The stress resultants are defined as follows:

$$\overset{(n)}{N}{}^{\alpha\beta} = \int_{-h/2}^{+h/2} s^{\alpha\beta}(x^3)^n \, \mathrm{d}x^3, \qquad \overset{(n)}{Q}{}^{\alpha} = \int_{-h/2}^{+h/2} s^{\alpha 3}(x^3)^n \, \mathrm{d}x^3, \qquad \overset{(n)}{T} = \int_{-h/2}^{+h/2} s^{33}(x^3)^n \, \mathrm{d}x^3. \tag{11}$$

For the equations of boundary conditions along the boundaries on the middle surface, the following quantities:

$$\overset{(n)}{v}_{\alpha} \quad \text{or} \quad v_{\beta}\left[\overset{(n)}{N}{}^{\alpha\beta} + \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \overset{(\ell)}{s}{}^{\beta\lambda}_{0} \overset{(m)}{v}{}^{\alpha}{}_{,\lambda} f(n+m+\ell+1) \right], \tag{12}$$

$$\overset{(n)}{v}_{3} \quad \text{or} \quad v_{\beta}\left[\overset{(n)}{Q}{}^{\beta} + \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \overset{(\ell)}{s}{}^{\alpha\beta}_{0} \overset{(m)}{v}{}^{3}{}_{,\alpha} f(n+m+\ell+1) \right], \tag{13}$$

are to be prescribed.

2.3. CONSTITUTIVE RELATIONS

For elastic and isotropic materials, the constitutive relations can be written as

$$s^{\alpha\beta} = D_{00}\delta^{\alpha\lambda}\delta^{\beta\nu}\gamma_{\lambda\nu} + E_1\delta^{\alpha\beta}(\gamma_3^3 + \delta^{\lambda\nu}\gamma_{\lambda\nu}), \tag{14}$$

$$s^{\alpha 3} = D_{00}\delta^{\alpha\lambda}\gamma_{\lambda}^{3}, \qquad s^{33} = D_{00}\gamma^{33} + E_1(\gamma_3^3 + \delta^{\lambda\nu}\gamma_{\lambda\nu}), \tag{15}$$

where $\delta^{\alpha\beta}$ is Kronecker's delta and Lamé's constants D_{00} and E_1 are defined by using Young's modulus E and the Poisson ratio v as follows:

$$D_{00} \equiv \frac{E}{1+v}, \qquad E_1 \equiv \frac{vE}{(1+v)(1-2v)}. \tag{16}$$

2.4. STRESS RESULTANTS IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

Stress resultants can be expressed in terms of the expanded displacement components as

$$\overset{(n)}{N}{}^{\alpha\beta} = \sum_{m=0}^{\infty} \left[\frac{D_{00}}{2} \delta^{\alpha\lambda}\delta^{\beta v}(\overset{(m)}{v}_{\lambda,v} + \overset{(m)}{v}_{v,\lambda} - 2b_{\lambda v}\overset{(m)}{v}_3) + E_1\delta^{\alpha\beta}\{(m+1)\overset{(m+1)}{v_3} + \tfrac{1}{2}\delta^{\lambda v}(\overset{(m)}{v}_{\lambda,v} + \overset{(m)}{v}_{v,\lambda} - 2b_{\lambda v}\overset{(m)}{v}_3)\} \right] f(n+m+1), \tag{17}$$

$$\overset{(n)}{Q}{}^{\alpha} = \sum_{m=0}^{\infty} \left[\frac{D_{00}}{2} \delta^{\alpha\lambda}\{(m+1)\overset{(m+1)}{v_\lambda} + \overset{(m)}{v}_{3,\lambda}\} \right] f(n+m+1), \tag{18}$$

$$\overset{(n)}{T} = \sum_{m=0}^{\infty} [(D_{00}+E_1)(m+1)\overset{(m+1)}{v_3} + \frac{E_1}{2}\delta^{\lambda v}(\overset{(m)}{v}_{\lambda,v} + \overset{(m)}{v}_{v,\lambda} - 2b_{\lambda v}\overset{(m)}{v}_3)\}] f(n+m+1), \tag{19}$$

where $n, m = 0, 1, 2, \ldots, \infty$.

2.5. EQUATIONS OF MOTION IN TERMS OF THE EXPANDED DISPLACEMENT COMPONENTS

The equations of motion can be expressed in terms of the expanded displacement components as

$$\delta\overset{(n)}{v}_\beta\colon \sum_{m=0}^{\infty} \left[[\tfrac{1}{2}(D_{00}\delta^{\alpha\lambda}\delta^{\beta v} + E_1\delta^{\alpha\beta}\delta^{\lambda v})(\overset{(m)}{v}_{\lambda,v} + \overset{(m)}{v}_{v,\lambda} - 2b_{\lambda v}\overset{(m)}{v}_3) + E_1\delta^{\alpha\beta}(m+1)\overset{(m+1)}{v_3}]_{,\alpha} - \rho\,\overset{(m)}{\ddot{v}}{}^{\beta}\} f(n+m+1) - \frac{n}{2} D_{00}\delta^{\beta\lambda}[(m+1)\overset{(m+1)}{v_\lambda} + \overset{(m)}{v}_{3,\lambda}] f(n+m) + \sum_{\ell=0}^{\infty} \overset{(\ell)}{s}{}_0^{\alpha\lambda}\,\overset{(m)}{v}{}^{\beta}{}_{,\alpha\lambda} f(n+m+\ell+1) \right] = 0, \tag{20}$$

$$\delta\overset{(n)}{v}_3\colon \sum_{m=0}^{\infty} \left[\{b_{\alpha\beta}[\tfrac{1}{2}(D_{00}\delta^{\alpha\lambda}\delta^{\beta v} + E_1\delta^{\alpha\beta}\delta^{\lambda v})(\overset{(m)}{v}_{\lambda,v} + \overset{(m)}{v}_{v,\lambda} - 2b_{\lambda v}\overset{(m)}{v}_3) + E_1\delta^{\alpha\beta}(m+1)\overset{(m+1)}{v_3}] + \frac{D_{00}}{2}\delta^{\alpha\lambda}[(m+1)\overset{(m+1)}{v_\lambda} + \overset{(m)}{v}_{3,\lambda}]_{,\alpha} - \rho\,\overset{(m)}{\ddot{v}}{}^{3}\} f(n+m+1) \right.$$

$$-n[(D_{00}+E_1)(m+1)\overset{(m+1)}{v_3}+\frac{E_1}{2}\delta^{\alpha\lambda}(\overset{(m)}{v}_{\alpha,\lambda}+\overset{(m)}{v}_{\lambda,\alpha}-2b_{\alpha\lambda}\overset{(m)}{v_3})]f(n+m)$$

$$\left.+\sum_{\ell=0}^{\infty}\overset{(\ell)}{s}{}_0^{\alpha\beta}\overset{(m)}{v}{}^3{}_{,\alpha\beta}f(n+m+\ell+1)\right]=0. \tag{21}$$

2.6. *M*TH ORDER APPROXIMATE THEORY

Since the fundamental equations mentioned above are complex, approximate theories of various orders may be considered for the present problem. A set of the following combination of displacement components for Mth ($M \geqslant 1$) order approximate equations is proposed by

$$v_\beta=\sum_{m=0}^{2M-1}\overset{(m)}{v_\beta}(x^3)^m, \qquad v_3=\sum_{m=0}^{2M-2}\overset{(m)}{v_3}(x^3)^m, \tag{22}$$

where $m = 0, 1, 2, 3, \ldots, M$.

The total number of the unknown displacement components is $(6M-1)$. In the above cases of $M=1$, an assumption of plane strains is inherently imposed. Another set of governing equations of the lowest order approximate theory ($M=1$) is derived with the use of an assumption that the normal stress s^{33} is zero which is known as a first order shear deformation theory (FST) with the shear correction coefficient $\kappa^2 = 1$ in the case of plates and shells.

Under the assumption of plane stresses, the shear strain $\gamma_{\alpha 3}$ must vanish through the thickness of a shell and the lowest order approximate theory reduces to the classical shell theory (CST).

3. FOURIER SERIES SOLUTION FOR A SIMPLY SUPPORTED SHALLOW SHELL

In the following, the Cartesian co-ordinates are used and expressed as $x \equiv x^1$, $y \equiv x^2$, $z \equiv x^3$ and the displacement components, $\overset{(n)}{u} \equiv \overset{(n)}{v_1}$, $\overset{(n)}{v} \equiv \overset{(n)}{v_2}$, $\overset{(n)}{w} \equiv \overset{(n)}{v_3}$.

The middle surface of a shallow shell as shown in Fig. 1 is described by

$$z=\frac{1}{2R_x}\left(x-\frac{a}{2}\right)^2+\frac{1}{R_{xy}}\left(x-\frac{a}{2}\right)\left(y-\frac{b}{2}\right)+\frac{1}{2R_y}\left(y-\frac{b}{2}\right)^2, \tag{23}$$

where a and b are reference lengths of the shell planform. R_x and R_y are the radii of curvature, in the x and y directions, respectively, and R_{xy} is the radius of torsion. The restrictive requirement in shallow shell theories that

$$(z_{,x})^2 \ll 1, \qquad z_{,x}z_{,y} \ll 1, \qquad (z_{,y})^2 \ll 1 \tag{24}$$

must be added.

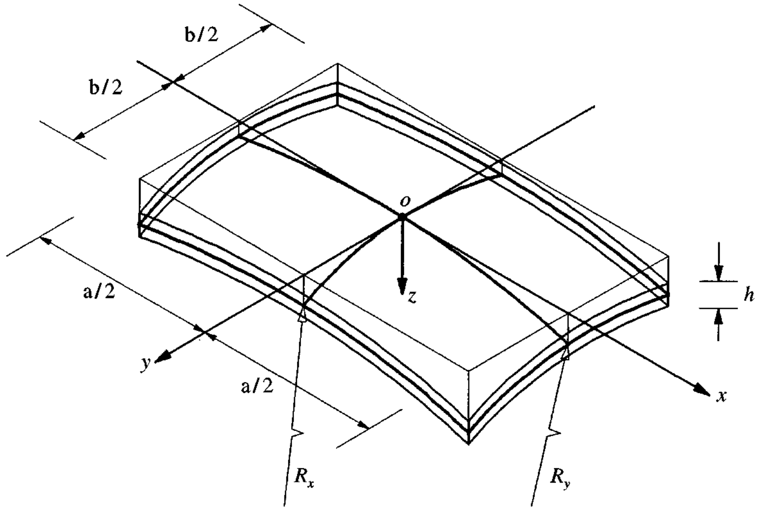


Figure 1. Co-ordinates and geometry of a shallow shell with rectangular planform.

The components of curvature tensor $b_{\alpha\beta}$ can be written as

$$b_{11} = z_{,xx} = \frac{1}{R_x}, \qquad b_{12} = b_{21} = z_{,xy} = \frac{1}{R_{xy}}, \qquad b_{22} = z_{,yy} = \frac{1}{R_y}. \tag{25}$$

For a doubly curved shallow shell of rectangular planform, the middle surface of a shell is assumed as follows:

$$\frac{1}{R_x} \neq 0, \qquad \frac{1}{R_{xy}} = 0 \tag{26}$$

and three types of shallow shells with positive, zero and negative Gaussian curvature $1/R_x R_y$ are considered in the present numerical examples.

Boundary conditions (12) and (13) for a simply supported shallow shell can be expressed on the x-constant edges,

$$\overset{(n)}{u}_{,x} = 0, \qquad \overset{(n)}{v} = 0, \qquad \overset{(n)}{w} = 0 \tag{27}$$

and on the y-constant edges,

$$\overset{(n)}{u} = 0, \qquad \overset{(n)}{v}_{,y} = 0, \qquad \overset{(n)}{w} = 0. \tag{28}$$

In the following analysis, the in-plane stresses are assumed to distribute uniformly in the depth direction. Only the first term of the expanded in-plane stress (7) is considered, i.e., $s_0^{\alpha\beta} = \overset{(0)}{s}{}_0^{\alpha\beta}$. In the present analysis, the following combination of the uniaxial ($\lambda = 0$) and biaxial ($\lambda = 1$) in-plane stresses is taken into consideration:

$$s_0^{yy} = \lambda s_0^{xx} \quad \text{and} \quad s_0^{xy} = s_0^{yx} = 0. \tag{29}$$

Following the Navier solution procedure, displacement components that satisfy the equations of boundary conditions (27) and (28) may be expressed as

$$\overset{(n)}{u} = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{u}_{rs} \cos\frac{r\pi x}{a} \sin\frac{s\pi y}{b} e^{i\omega t}, \qquad \overset{(n)}{v} = \sum_{r=1}^{\infty}\sum_{s=0}^{\infty} \overset{(n)}{v}_{rs} \sin\frac{r\pi x}{a} \cos\frac{s\pi y}{b} e^{i\omega t}, \tag{30}$$

$$\overset{(n)}{w} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \overset{(n)}{w}_{rs} \sin\frac{r\pi x}{a} \sin\frac{s\pi y}{b} e^{i\omega t}, \tag{31}$$

where r and s are the displacement mode numbers, ω denotes the circular frequency and i, the imaginary unit.

The equations of motion are rewritten in terms of the generalized displacement components, $\overset{(n)}{u}_{rs}$, $\overset{(n)}{v}_{rs}$ and $\overset{(n)}{w}_{rs}$. The dimensionless natural frequency Ω and in-plane stress Λ in the x direction are defined as follows:

$$\Omega = \omega h\sqrt{\rho/G}, \qquad G = E/2(1+\nu), \tag{32}$$

$$\Lambda = h s_0^{xx} b^2/\pi^2 D, \qquad D = Eh^3/12(1-\nu^2). \tag{33}$$

4. EIGENVALUE PROBLEM OF A THICK SHALLOW SHELL

Equations (20) and (21) can be rewritten by collecting the coefficients for the generalized displacements of any fixed values r and s. The generalized displacement vector $\{\mathbf{U}\}$ for the Mth order approximate theory is expressed as

$$\{\mathbf{U}\}^{\mathrm{T}} = \{\overset{(0)}{u}_{rs}, \ldots, \overset{(2M-1)}{u_{rs}}; \overset{(0)}{v}_{rs}, \ldots, \overset{(2M-1)}{v_{rs}}; \overset{(0)}{w}_{rs}, \ldots, \overset{(2M-2)}{w_{rs}}\}. \tag{34}$$

For free vibration problems, the equations of motion can be expressed as the following eigenvalue problem:

$$([\mathbf{K}] - \Omega^2[\mathbf{M}])\{\mathbf{U}\} = 0, \tag{35}$$

where matrix $[\mathbf{K}]$ denotes the stiffness matrix which may contain the terms of the in-plane stresses, and matrix $[\mathbf{M}]$, the mass matrix.

For stability problems, the natural frequency vanishes and the stability equation can be expressed as the following eigenvalue problem:

$$([\mathbf{K}] + \Lambda[\mathbf{S}])\{\mathbf{U}\} = 0, \tag{36}$$

where matrix $[\mathbf{K}]$ denotes the stiffness matrix, and matrix $[\mathbf{S}]$, the geometric-stiffness matrix due to the in-plane stresses.

The power method [15] is used to obtain the numerical solution of the eigenvalue problems. Although all the eigenvalues and eigenvectors can be computed by this method for each deformation mode of r and s, the dominant eigenvalues which correspond to the lower natural frequencies and critical buckling stresses are of most concern.

5. NUMERICAL EXAMPLES AND RESULTS

5.1. NUMERICAL EXAMPLES

Natural frequencies and buckling stresses of a thick elastic shallow shell with simply supported square edges ($a = b$) are analyzed for six numerical examples with the thickness parameter

$$a/h = 1,\ 2,\ 4,\ 5,\ 10,\ 20. \tag{37}$$

Since the thickness–curvature ratio is assumed to be $h/R \ll 1$, the limit of this parameter is taken to be $h/R = 0{\cdot}2$. The curvature parameters a/R_x and b/R_y are varied from 0 to $\pm 0{\cdot}4$ for three types of shallow shells, i.e., spherical, cylindrical and hyperbolic parabolidal shells. The Poisson ratio is fixed to be $\nu = 0{\cdot}3$. All the numerical results are shown in the dimensionless quantities.

Although the present sets of approximate theories of any order can easily be applied to a moderately thick shell, higher orders of the expanded two-dimensional theories may be necessary to obtain reasonably accurate solutions for an extremely thick shell. It is noticed that the proper order of present approximate theories may be estimated according to the level of thickness parameters of the shell.

Only the first term of the expanded in-plane stress in equation (7) is considered and the natural frequencies of a thick shallow shell subjected to uniaxial and biaxial in-plane tensile/compressive stresses are shown in the present examples.

5.2. CONVERGENCE OF THE FIRST TWO NATURAL FREQUENCIES AND COMPARISON WITH THOSE OF EXISTING THEORIES

In order to verify the accuracy of the present solutions, the convergence properties of the first two natural frequencies Ω_1 and Ω_2 of shallow shells without in-plane stresses for the displacement mode $r = s = 1$ are shown in Table 1. They are different from the case of plates that the two types of flexural and extensional displacement modes are not separated from each other. The lower natural frequency Ω_1 is predominantly flexural modes with some shear deformations, whereas the upper frequency Ω_2 is predominantly extensional modes with thickness changes. The numerical accuracy of the natural frequencies of plates has been shown in a previous paper [5]. Since there is no data available to compare with the present

TABLE 1

Convergence of the first two natural frequencies Ω_1, Ω_2 for $r = s = 1$ and comparison with other results ($a = b$)

a/h	a/R_x	b/R_y	CST	FST	M 1	M 2	M 3	M 4
(a) Ω_1: *Predominantly flexural mode (first mode)*								
2	0·0	0·0	2·0270	1·4939	1·5597	1·5185	1·5158	←
	0·2	0·2	2·0280	1·4991	1·5643	1·5221	1·5191	←
		0·0	2·0189	1·4927	1·5580	1·5168	1·5140	←
		−0·2	1·9973	1·4839	1·5487	1·5082	1·5054	←
	0·4	0·4	2·0312	1·5143	1·5777	1·5326	1·5290	←
		0·0	1·9983	1·4891	1·5532	1·5117	1·5088	←
		−0·4	1·9364	1·4563	1·5184	1·4795	1·4769	←
5	0·0	0·0	0·3732	0·3406	0·3454	0·3421	←	←
	0·2	0·2	0·3777	0·3458	0·3505	0·3470	←	←
		0·0	0·3739	0·3415	0·3463	0·3430	0·3429	←
		−0·2	0·3715	0·3390	0·3439	0·3406	0·3405	←
	0·4	0·4	0·3909	0·3607	0·3652	0·3611	0·3610	←
		0·0	0·3760	0·3442	0·3490	0·3454	←	←
		−0·4	0·3665	0·3346	0·3394	0·3361	←	←
(b) Ω_2: *Predominantly extensional mode (second mode)*								
2	0·0	0·0	2.2214	←	←	←	←	←
	0·2	0·2	2·2214	←	←	←	←	←
		0·0	2·2306	2·2252	2·2254	2·2253	←	←
		−0·2	2·2545	2·2362	2·2370	2·2365	←	←
	0·4	0·4	2·2214	←	←	←	←	←
		0·0	2·2545	2·2362	2·2370	2·2365	←	←
		−0·4	2·3253	2·2782	2·2810	2·2794	2·2793	←
5	0·0	0·0	0·8886	←	←	←	←	←
	0·2	0·2	0·8886	←	←	←	←	←
		0·0	0·8896	←	←	←	←	←
		−0·2	0·8927	0·8926	←	←	←	←
	0·4	0·4	0·8886	←	←	←	←	←
		0·0	0·8927	0·8926	←	←	←	←
		−0·4	0·9047	0·9045	←	←	←	←

Note: CST: Classical shallow shell theory (including the rotatory inertia).
FST: First order shear deformation theory ($\kappa^2 = 5/6$).
$M = 1$: $s^{33} = 0$ [first order shear deformation theory ($\kappa^2 = 1$)].

results of simply supported thick shallow shells, a direct comparison of the present solutions with those of the classical shallow shell theory (CST) in which the effect of extension and rotatory inertia are included is made. The present natural frequencies are also compared with the result based on a first order shear deformation theory (FST) which corresponds to the Mindlin plate theory in which a shear correction factor κ^2 is introduced to correct the contradictory shear stress distribution over the thickness of the shell. It is noticed that the proper order of the present approximate theories may be estimated according to the level of a/h and a/R_x. Since the present results for $M = 1 - 3$ converge accurately enough within the present order of approximate theories, only the more accurate numerical results for $M = 5$ are discussed in the following.

5.3. NATURAL FREQUENCIES WITHOUT IN-PLANE STRESSES

The first nine natural frequencies of shallow shells without in-plane stresses are shown in Table 2 for all values of a/h and selected curvature parameters of a/R_x and b/R_y within the assumption of the shallow shell theory. The mode classification number of three figures on the right shoulder of natural frequencies in Table 2 defines the vibration mode. The first and second figures denote the mode numbers of r and s, respectively, and the last figure, the mode order number which corresponds to displacement distributions in the thickness direction, i.e., "123", shows the third vibration mode for $r = 1$, $s = 2$. For thin shells, as shown in the results from the classical shell theory, only the lowest frequencies for each mode number of r and s appear in the set of the first nine frequencies. However, for thicker shells, higher frequencies for a lower mode number of r and s will appear in accordance with the thickness parameter a/h.

In the case of a plate with the thickness parameter $a/h = 10$, the present results agree with the exact values of the three-dimensional elasticity theory [16]. Natural frequencies of the in-plane shear mode agree with the numerical solutions of a combined finite-element and Rayleigh–Ritz method [17]. For very thick plates with the thickness parameter $a/h = 2$ and 5, the present results agree with recent results yielded by a continuum three-dimensional Ritz formulation [6].

5.4. NATURAL FREQUENCIES VERSUS IN-PLANE STRESSES CURVES AND BUCKLING STRESSES

In Figures 2(a,b), the variation of the lowest two natural frequencies for $r = s = 1$ with respect to in-plane stresses is shown for $a/h = 2$ and 5. Figure 2(a) shows the results for uniaxial in-plane stresses ($\lambda = 0$) and Figure 2(b), for biaxial ones ($\lambda = 1$). The open circle and square show the natural frequencies which correspond to predominantly flexural (Ω_1) and extensional (Ω_2) modes with some shear deformation and thickness change respectively. The second natural frequency Ω_2 is nearly constant for all the values of in-plane stresses. When the lowest natural frequency Ω_1 vanishes, the in-plane stresses reduce to the critical buckling stresses of the shallow shells. The first frequency curve (Ω_1) will decrease rapidly prior to buckling and the frequency vanishes at the in-plane buckling stress.

The buckling stresses can be calculated usually through the stability equation (36) as eigenvalue problems. In the case of a simply supported shallow shell subjected to arbitrary in-plane stresses Λ, the natural frequency Ω_a can be expressed explicitly with reference to the natural frequency Ω_0 of a shell without in-plane stresses. The relation between Ω_a and Ω_0 can be obtained from a comparison of the

TABLE 2

The first nine natural frequencies ($a = b$)

			Mode number								
a/h	a/R_x	b/R_y	1	2	3	4	5	6	7	8	9
1	0·0	0·0	$3\cdot1416^{101}$	$3\cdot1416^{011}$	$3\cdot7419^{111}$	$4\cdot4429^{102}$	$4\cdot4429^{012}$	$4\cdot4429^{112}$	$5\cdot2013^{113}$	$5\cdot4414^{114}$	$6\cdot2832^{201}$
	0·1	0·1	$3\cdot1416^{101}$	$3\cdot1416^{011}$	$3\cdot7414^{111}$	$4\cdot4429^{102}$	$4\cdot4429^{012}$	$4\cdot4429^{112}$	$5\cdot2023^{113}$	$5\cdot4414^{114}$	$6\cdot2832^{201}$
		0·0	$3\cdot1416^{101}$	$3\cdot1416^{011}$	$3\cdot7387^{111}$	$4\cdot4429^{102}$	$4\cdot4429^{012}$	$4\cdot4465^{112}$	$5\cdot1970^{113}$	$5\cdot4464^{114}$	$6\cdot2832^{201}$
		−0·1	$3\cdot1416^{101}$	$3\cdot1416^{011}$	$3\cdot7298^{111}$	$4\cdot4429^{102}$	$4\cdot4429^{012}$	$4\cdot4572^{112}$	$5\cdot1840^{113}$	$5\cdot4604^{114}$	$6\cdot2832^{201}$
	0·2	0·2	3.1416^{101}	$3\cdot1416^{011}$	$3\cdot7399^{111}$	$4\cdot4429^{102}$	$4\cdot4429^{012}$	$4\cdot4429^{112}$	$5\cdot2052^{113}$	$5\cdot4414^{114}$	$6\cdot2832^{201}$
		0·0	3.1416^{101}	$3\cdot1416^{011}$	$3\cdot7294^{111}$	$4\cdot4429^{102}$	$4\cdot4429^{012}$	$4\cdot4571^{112}$	$5\cdot1849^{113}$	$5\cdot4604^{114}$	6.2832^{201}
		−0·2	3.1416^{101}	$3\cdot1415^{011}$	$3\cdot6963^{111}$	$4\cdot4429^{102}$	$4\cdot4429^{012}$	$4\cdot4974^{112}$	$5\cdot1424^{113}$	$5\cdot5066^{114}$	6.2832^{201}
2	0·0	0·0	1.5158^{111}	$1\cdot5708^{101}$	$1\cdot5708^{011}$	$2\cdot2214^{112}$	$2\cdot8066^{211}$	$2\cdot8066^{121}$	$3\cdot1416^{201}$	$3\cdot1416^{021}$	3.5124^{102}
	0·2	0·2	$1\cdot5191^{111}$	$1\cdot5708^{101}$	$1\cdot5708^{011}$	$2\cdot2214^{112}$	$2\cdot8070^{211}$	$2\cdot8070^{121}$	$3\cdot1416^{201}$	$3\cdot1416^{021}$	$3\cdot5124^{102}$
		0·0	$1\cdot5140^{111}$	$1\cdot5708^{101}$	$1\cdot5708^{011}$	$2\cdot2253^{112}$	$2\cdot8036^{211}$	$2\cdot8068^{121}$	$3\cdot1416^{201}$	$3\cdot1416^{021}$	$3\cdot5124^{102}$
		−0·2	$1\cdot5054^{111}$	$1\cdot5708^{101}$	$1\cdot5708^{011}$	$2\cdot2365^{112}$	$2\cdot8007^{211}$	$2\cdot8007^{121}$	$3\cdot1416^{201}$	$3\cdot1416^{021}$	$3\cdot5124^{102}$
	0·4	0·4	$1\cdot5290^{111}$	$1\cdot5708^{101}$	$1\cdot5708^{011}$	$2\cdot2214^{112}$	$2\cdot8079^{211}$	$2\cdot8079^{121}$	$3\cdot1416^{201}$	$3\cdot1416^{021}$	$3\cdot5124^{102}$
		0·0	$1\cdot5088^{111}$	$1\cdot5708^{101}$	$1\cdot5708^{011}$	$2\cdot2365^{112}$	$2\cdot7948^{211}$	$2\cdot8073^{121}$	$3\cdot1416^{201}$	$3\cdot1416^{021}$	$3\cdot5124^{102}$
		−0·4	1.4769^{111}	$1\cdot5708^{101}$	$1\cdot5708^{011}$	$2\cdot2793^{112}$	$2\cdot7838^{211}$	$2\cdot7838^{121}$	$3\cdot1416^{201}$	$3\cdot1416^{021}$	$3\cdot5124^{102}$
4	0·0	0·0	$0\cdot5066^{111}$	$0\cdot7854^{101}$	$0\cdot7854^{011}$	$1\cdot0692^{211}$	$1\cdot0692^{121}$	$1\cdot1107^{112}$	$1\cdot5158^{221}$	$1\cdot5708^{201}$	$1\cdot5708^{021}$
	0·2	0·2	$0\cdot5113^{111}$	$0\cdot7854^{101}$	$0\cdot7854^{011}$	$1\cdot0708^{211}$	$1\cdot0708^{121}$	$1\cdot1107^{112}$	$1\cdot5166^{221}$	$1\cdot5708^{201}$	$1\cdot5708^{021}$
		0·0	$0\cdot5071^{111}$	$0\cdot7854^{101}$	$0\cdot7854^{011}$	$1\cdot0685^{211}$	$1\cdot0704^{121}$	$1\cdot1121^{112}$	$1\cdot5153^{221}$	$1\cdot5708^{201}$	$1\cdot5708^{021}$
		−0·2	$0\cdot5041^{111}$	$0\cdot7854^{101}$	$0\cdot7854^{011}$	$1\cdot0686^{211}$	$1\cdot0686^{121}$	$1\cdot1160^{112}$	$1\cdot5131^{221}$	$1\cdot5708^{201}$	$1\cdot5708^{021}$
	0·4	0·4	$0\cdot5249^{111}$	$0\cdot7854^{101}$	$0\cdot7854^{011}$	$1\cdot0756^{211}$	$1\cdot0756^{121}$	$1\cdot1107^{112}$	$1\cdot5191^{221}$	$1\cdot5708^{201}$	$1\cdot5708^{021}$
		0·0	$0\cdot5088^{111}$	$0\cdot7854^{101}$	$0\cdot7854^{011}$	$1\cdot0665^{211}$	$1\cdot0740^{121}$	$1\cdot1160^{112}$	$1\cdot5140^{221}$	$1\cdot5708^{201}$	$1\cdot5708^{021}$
		−0·4	$0\cdot4971^{111}$	$0\cdot7854^{101}$	$0\cdot7854^{011}$	$1\cdot0668^{211}$	$1\cdot0668^{121}$	$1\cdot1317^{112}$	$1\cdot5054^{221}$	$1\cdot5708^{201}$	$1\cdot5708^{021}$

TABLE 2 (*Continued*)

The first nine natural frequencies ($a = b$)

a/h	a/R_x	b/R_y	Mode number 1	2	3	4	5	6	7	8	9
5	0·0	0·0	0·3421^{111}	0·6283^{101}	0·6283^{011}	0·7511^{211}	0·7511^{121}	0·8886^{112}	1·0889^{221}	1·2566^{201}	1·2566^{021}
	0·2	0·2	0·3470^{111}	0·6283^{101}	0·6283^{011}	0·7529^{211}	0·7529^{121}	0·8886^{112}	1·0898^{221}	1·2566^{201}	1·2566^{021}
		0·0	0·3429^{111}	0·6283^{101}	0·6283^{011}	0·7507^{211}	0·7524^{121}	0·8896^{112}	1·0887^{221}	1·2566^{201}	1·2566^{021}
		−0·2	0·3405^{111}	0·6283^{101}	0·6283^{011}	0·7510^{211}	0·7510^{121}	0·8926^{112}	1·0872^{221}	1·2566^{201}	1·2566^{021}
	0·4	0·4	0·3610^{111}	0·6283^{101}	0·6283^{011}	0·7581^{211}	0·7581^{121}	0·8886^{112}	1·0928^{221}	1·2566^{201}	1·2566^{021}
		0·0	0·3454^{111}	0·6283^{101}	0·6283^{011}	0·7494^{211}	0·7561^{121}	0·8926^{112}	1·0882^{221}	1·2566^{201}	1·2566^{021}
		−0·4	0·3360^{111}	0·6283^{101}	0·6283^{011}	0·7507^{211}	0·7507^{121}	0·9045^{112}	1·0824^{221}	1·2566^{201}	1·2566^{021}
10	0·0	0·0	0·09315^{111}	0·2226^{211}	0·2226^{121}	0·3142^{101}	0·3142^{011}	0·3421^{221}	0·4171^{311}	0·4171^{131}	0·4443^{112}
	0·2	0·2	0·09826^{111}	0·2246^{211}	0·2246^{121}	0·3142^{101}	0·3142^{011}	0·3433^{221}	0·4181^{311}	0·4181^{131}	0·4443^{112}
		0·0	0·09436^{111}	0·2226^{211}	0·2239^{121}	0·3142^{101}	0·3142^{011}	0·3423^{221}	0·4170^{311}	0·4180^{131}	0·4448^{112}
		−0·2	0·09276^{111}	0·2232^{211}	0·2232^{121}	0·3142^{101}	0·3142^{011}	0·3417^{221}	0·4177^{311}	0·4177^{131}	0·4461^{112}
	0·4	0·4	0·1120^{111}	0·2306^{211}	0·2306^{121}	0·3142^{101}	0·3142^{011}	0·3470^{221}	0·4210^{311}	0·4210^{131}	0·4443^{112}
		0·0	0·09785^{111}	0·2224^{211}	0·2279^{121}	0·3142^{101}	0·3142^{011}	0·3429^{221}	0·4167^{311}	0·4207^{131}	0·4461^{112}
		−0·4	0·09163^{111}	0·2248^{211}	0·2248^{121}	0·3142^{101}	0·3142^{011}	0·3405^{221}	0·4196^{311}	0·4196^{131}	0·4516^{112}
20	0·0	0·0	0·02387^{111}	0·05893^{211}	0·05893^{121}	0·09315^{221}	0·1155^{311}	0·1155^{131}	0·1485^{321}	0·1485^{231}	0·1571^{101}
	0·2	0·2	0·02872^{111}	0·06102^{211}	0·06102^{121}	0·09446^{221}	0·1166^{311}	0·1166^{131}	0·1492^{321}	0·1492^{231}	0·1571^{101}
		0·0	0·02515^{111}	0·05898^{211}	0·06029^{121}	0·09345^{221}	0·1155^{311}	0·1164^{131}	0·1485^{321}	0·1488^{231}	0·1571^{101}
		−0·2	0·02378^{111}	0·05965^{211}	0·05965^{121}	0·09305^{221}	0·1162^{311}	0·1162^{131}	0·1485^{321}	0·1485^{231}	0·1571^{101}
	0·4	0·4	0·03975^{111}	0·06686^{211}	0·06686^{121}	0·09826^{221}	0·1196^{311}	0·1196^{131}	0·1516^{321}	0·1516^{231}	0·1571^{101}
		0·0	0·02861^{111}	0·05914^{211}	0·06418^{121}	0·09436^{221}	0·1154^{311}	0·1190^{131}	0·1486^{321}	0·1500^{231}	0·1571^{101}
		−0·4	0·02349^{111}	0·06173^{211}	0·06173^{121}	0·09276^{221}	0·1182^{311}	0·1182^{131}	0·1486^{321}	0·1486^{231}	0·1571^{101}

equations of motion as follows:

$$\Omega_a^2 = \Omega_0^2 + \frac{\pi^2}{6(1-v)}\left(\frac{h}{b}\right)^2 (\alpha^2 + \lambda\beta^2)\Lambda, \tag{38}$$

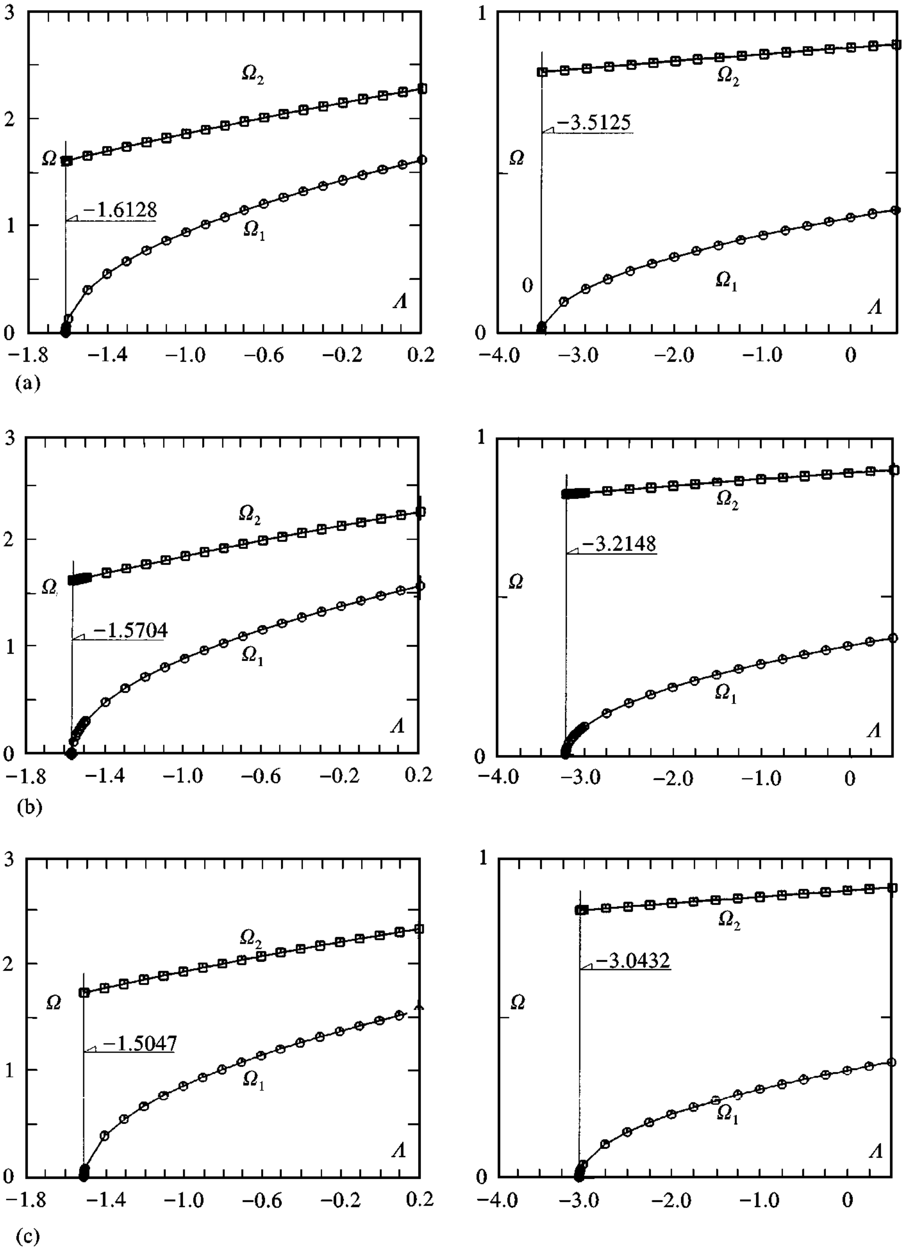


Figure 2(a). The first two natural frequencies Ω_1, Ω_2 for $r = s = 1$ versus uniaxial in-plane stress Λ ($\lambda = 0$; $a = b$; left side: $a/h = 2$; right side: $a/h = 5$). (a) Spherical shell, $a/R_x = b/R_y = 0{\cdot}4$. (b) Cylindrical shell, $a/R_x = 0{\cdot}4$, $b/R_y = 0{\cdot}0$. (c) Hyperbolic paraboloidal shell, $a/R_x = -\,b/R_y = 0{\cdot}4$.

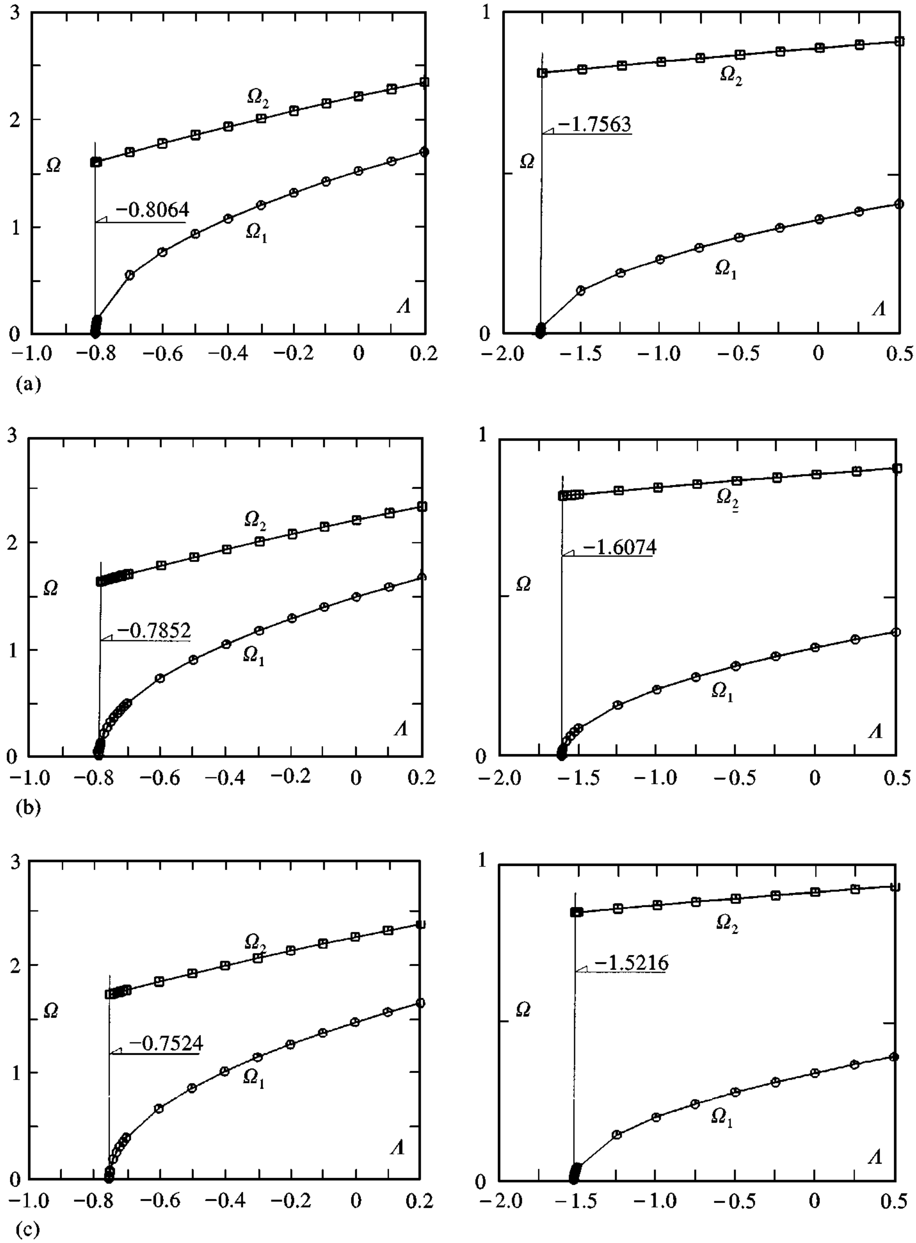


Figure 2(b). The first two natural frequencies Ω_1, Ω_2 for $r = s = 1$ versus biaxial in-plane stress Λ ($\lambda = 1$; $a = b$; left side: $a/h = 2$; right side: $a/h = 5$). (a) Spherical shell, $a/R_x = b/R_y = 0{\cdot}4$. (b) Cylindrical shell, $a/R_x = 0{\cdot}4$, $b/R_y = 0{\cdot}0$. (c) Hyperbolic paraboloidal shell, $a/R_x = -\,b/R_y = 0{\cdot}4$.

where

$$\alpha = r\pi\left(\frac{h}{a}\right), \qquad \beta = s\pi\left(\frac{h}{b}\right). \tag{39}$$

When the natural frequency Ω_a vanishes under the in-plane stresses, elastic buckling occurs and the critical buckling stress Λ_{cr} relates with the natural frequency Ω_0 as

$$\Lambda_{cr} = -\frac{6(1-\nu)}{\pi^2}\left(\frac{b}{h}\right)^2 \frac{1}{(\alpha^2+\lambda\beta^2)}\Omega_0^2. \tag{40}$$

The critical buckling stresses of simply supported shallow shells subjected to in-plane stresses can be predicted from the natural frequency of the shell without in-plane stresses. The calculated critical buckling stresses of simply supported square shells under in-plane compressions are shown in Table 3 for the first vibration mode $r = s = 1$.

5.5. EFFECT OF CURVATURES ON NATURAL FREQUENCIES

The effects of curvatures or shell configurations on the natural frequencies of shells are very interesting in comparison with plates. As seen in Table 2, these effects are not so remarkable within the range of slightly curved shallow shells. A qualitative representation may be seen in Figure 3. The difference distributions of the natural frequencies between shells and plates with square planform for the first two natural frequencies ($\Omega^S_{1,2} - \Omega^P_{1,2}$) of $r = s = 1$ are plotted in the figures. The lower (first) natural frequency $\Omega_1(111)$ corresponds to a predominantly flexural mode with shear deformations, whereas the upper (second) frequency $\Omega_2(112)$ corresponds to a predominantly extensional mode with thickness changes. The curvature parameter a/R_x varies from 0·0 to $+$ 0·4 and b/R_y, from $-$ 0·4 to $+$ 0·4 for shallow shells with the thickness parameter $a/h = 2$ and 5.

6. CONCLUSIONS

Natural frequencies of thick shallow shells calculated by using the classical shallow shell theory are usually overpredicted. In order to analyze the complete effects of higher order deformations on the natural frequencies of thick shallow shells, various orders of the expanded approximate shallow shell theories have been presented. The natural frequencies have been calculated for three types of simply supported thick shallow shells with positive, zero and negative Gaussian curvatures. It has been shown that shear deformations and thickness changes have an important effect on the natural frequencies of thick shallow shells with and/or without in-plane stresses.

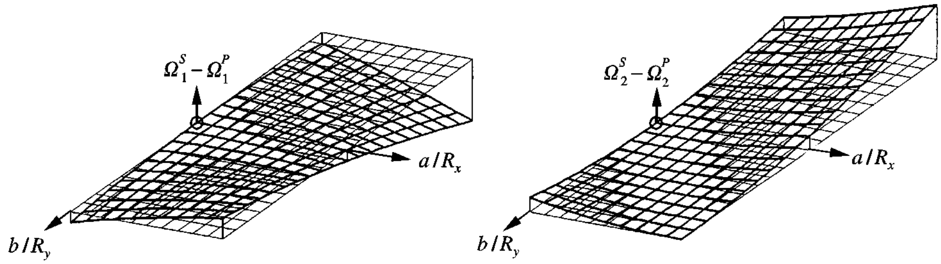
The following conclusions may be drawn from the present analysis:

(1) In order to verify the accuracy of the present results, the convergence properties of the numerical solutions according to the order of approximate theories have been examined. The numerical convergence of the first two natural frequencies for simply supported shallow shells without in-plane stresses has been examined in detail. The first nine natural frequencies of simply supported shallow shells without in-plane stresses have been obtained for all the values of a/h and several displacement modes. The present results obtained for $M = 5$ are considered to be accurate enough for extremely thick shallow shells with small a/h. It is noted

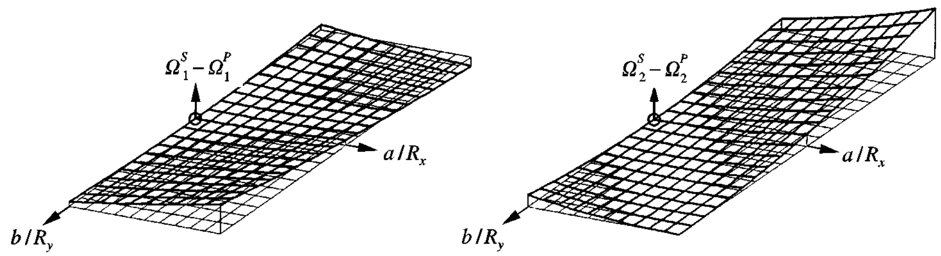
Table 3

Buckling stresses calculated from the natural frequencies without in-plane stresses ($a/b = 1$, $r = s = 1$, $\nu = 0{\cdot}3$)

a/h	a/R_x	b/R_y	Ω_0	Λ	
				$\lambda = 0$	$\lambda = 1$
1	0·0	0·0	3·7419	0·6037	0·3019
	0·1	0·1	3·7414	0·6036	0·3018
		0·0	3·7387	0·6027	0·3013
		−0·1	3·7298	0·5998	0·2999
	0·2	0·2	3·7399	0·6031	0·3015
		0·0	3·7294	0·5997	0·2999
		−0·2	3·6963	0·5891	0·2946
2	0·0	0·0	1·5158	1·5851	0·7925
	0·2	0·2	1·5191	1·5920	0·7960
		0·0	1·5140	1·5813	0·7907
		−0·2	1·5054	1·5634	0·7817
	0·4	0·4	1·5290	1·6128	0·8064
		0·0	1.5088	1·5705	0·7852
		−0·4	1·4769	1·5048	0·7524
4	0·0	0·0	0·5066	2·8328	1·4164
	0·2	0·2	0·5113	2·8856	1·4428
		0·0	0·5071	2·8384	1·4192
		−0·2	0·5041	2·8049	1·4025
	0·4	0·4	0·5249	3·0412	1·5206
		0·0	0·5088	2·8575	1·4287
		−0·4	0·4971	2·7276	1·3638
5	0·0	0·0	0·3421	3·1538	1·5769
	0·2	0·2	0·3470	3·2448	1·6224
		0·0	0·3429	3·1686	1·5843
		−0·2	0·3405	3·1244	1·5622
	0·4	0·4	0·3610	3·5119	1·7560
		0·0	0·3454	3·2150	1·6075
		−0·4	0·3360	3·0423	1·5212
10	0·0	0·0	0·09315	3·7412	1·8706
	0·2	0·2	0·09826	4·1630	2·0815
		0·0	0·09436	3·8391	1·9195
		−0·2	0·09276	3·7100	1·8550
	0·4	0·4	0·1120	5·4086	2·7043
		0·0	0·09785	4·1283	2·0642
		−0·4	0·09163	3·6201	1·8101
20	0·0	0·0	0·02387	3·9307	1·9654
	0·2	0·2	0·02872	5·6904	2·8452
		0·0	0·02515	4·3636	2·1818
		−0·2	0·02378	3·9012	1·9506
	0·4	0·4	0·03975	10·9004	5·4502
		0·0	0·02861	5·6468	2·8234
		−0·4	0·02349	3·8066	1·9033



(i) $a/h = 2$



(ii) $a/h = 5$

Figure 3. Difference distribution of the first two natural frequencies between shells and plates ($\Omega^S_{1,2} - \Omega^P_{1,2}$) for $r = s = 1$ with respect to curvature parameters [$0{\cdot}0 \leqslant a/R_x \leqslant +0{\cdot}4$, $-0{\cdot}4 \leqslant b/R_y \leqslant +0{\cdot}4$; $a = b$; left side: Ω_1 (111); right side: Ω_2 (112)].

that the two-dimensional higher order shallow shell theories in the present paper can predict the natural frequencies of a simply supported thick shallow shell accurately.

(2) For the lowest mode $r = s = 1$, the first two natural frequencies of simply supported shallow shells subjected to uniaxial and biaxial in-plane stresses have been obtained for all the values of a/h and several displacement modes. When the lowest natural frequency vanishes, the in-plane stresses reduce to the critical buckling stresses of shallow shells. It is necessary to take into account the complete effects of higher order deformations such as shear deformations and thickness changes for the analysis of vibration and stability problems of thick shallow shells.

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